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ON LEFT O-PRIME IDEALS OVER A NONCOMMUTATIVE RING

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Abstract. In this paper, we focus on a one-sided generalization of the concept of prime ideal in a noncommutative ring, which is called a left O-prime ideal. Some of its basic properties are investigated, pointing out both similarities and differences between left O-prime ideals and their commutative counterparts. Mainly, we prove a noncommutative generalization of Cohen's Theorem for left O-prime ideals and that any left ideal in R is the intersection of a finite number of left O-prime ideals of a noncommutative ring R satisfying the ascending chain condition on left O-radical ideals.

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1. Introduction

As is well-known, prime ideals form an important part in the commutative ring theory. Basically, Cohen's and Kaplansky's Theorems about prime ideals in commutative ring theory are useful to characterize the rings ([16],[17]). While there are many reasons why this is so, in this paper we will focus on the fact that left O-prime ideals control the structure of noncommutative rings. It is also well-known that the set of nilpotent elements of a commutative ring forms an ideal coinciding with the intersection of all the prime ideals; in noncommutative ring theory, however, the set of nilpotent elements need not form an ideal and the intersection of prime ideals of a ring is characterized by using m-system in [10]. A nonempty set $S \subseteq R$ is called an m-system if, for any a,$b \in S$, there exists $r \in R$ such that $arb \in S$. Then for any ideal I of any ring R, it follows that

$$rad_R(I) = \{s \in R : \text{every } m\text{-system containing } s \text{ meets } I\}.$$

With this motivation, in this paper, we define new concepts for a left ideal I of a ring R which are generalization of prime ideals, the radical of an ideal and nilpotent elements of a ring. Then we study properties of these concepts and relations among them. Let P be a left ideal of R. Then P is called a left O-prime ideal if for any left ideals I, J such that $PJ \subseteq P$ and $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$ holds. We give an example of a left ideal which is a left O-prime ideal but not a prime ideal of a ring R. Then we show that every maximal left ideal of a ring is a left O-prime ideal. For a left ideal I of R and the set $K = \{a_i \in R : a_0 = a$ and $a_{i+1} \in a_iRa_i, i \in \mathbb{N}\}$ of R such that $I \cap K = \varnothing$, we verify that there is a left O-prime ideal P of R containing I such that $P \cap K = \varnothing$. By using this result and under some conditions, we characterize elements of $O_R(I)$, the intersection of left O-prime ideals of R containing I. Moreover, we prove that any left O-radical ideal K (i.e.

$O_R(K) = K$) in R is the intersection of a finite number of left O-prime ideals if R satisfies the ascending chain condition on left O-radical ideals.

2. The O-radical of a Left Ideal

The concept of a prime ideal in commutative ring extend to two generalizations (prime ideal and completely prime ideal) for noncommutative ring theory; an ideal P of a ring R is called prime (completely prime) if either I or J (a or b) in P whenever $IJ \subseteq P$ ($ab \in P$) for ideals I,J of R ($a, b \in R$). In [3], R.L. Reyes introduced completely prime right ideals as a one-sided generalization of completely prime ideal in noncommutative ring and investigated some properties of this class. In [3], a right ideal P is a completely prime right ideal if for any $a, b \in P$ with $aP \subseteq P$, $ab \in P$ implies that either $a \in P$ or $b \in P$. Since the notion of prime ideal in noncommutive ring is more useful, now we give a generalization of a prime ideal.

Throughout the paper, R will denote a ring with identity.

Definition 2.1. *A left ideal P of a ring is said to be a left O-prime ideal if for any left ideals I, J such that $PJ \subseteq P$ and $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$ holds.*

It is clear that the notion is equivalent to the concept of prime ideal whenever P is an ideal. Let P be a left ideal of R. $\mathbb{I}_R(P)$ is the sum of left ideals J of R such that PJ is in P. Clearly, $\mathbb{I}_R(P)$ is a left ideal of R. Moreover, P is an ideal of R if and only if $\mathbb{I}_R(P) = R$.

Lemma 2.2. *Let P be a left ideal of R which is not right. Then P is a left O-prime ideal if and only if $\mathbb{I}_R(P) = P$.*

Proof. It is enough to show that $\mathbb{I}_R(P) = P$ for the completion. Then by the hypothesis, we get $\mathbb{I}_R(P) \neq R$. Take $a \in R \backslash \mathbb{I}_R(P)$ and $b \in \mathbb{I}_R(P)$. Hence there are elements $p \in P$ and $x \in R$ such that $c = pxa \notin P$. Thus $cRb \subseteq PRb \subseteq P$. Since P is a left O-prime ideal of R, we get either $c \in P$ or $b \in P$ and so we get that $b \in P$. Then $\mathbb{I}_R(P) = P$. □

Lemma 2.3. *Let I be a left ideal of R. Then*
i) $\mathbb{I}_R(I) \subseteq \mathbb{I}_R(I^2)$,
ii) $\mathbb{I}_R(I) \subseteq \mathbb{I}_R(\mathbb{I}_R(I))$,
iii) If $f : R \to S$ is a ring epimorphism, then $f(\mathbb{I}_R(I)) \subseteq \mathbb{I}_S(f(I))$.
If $Kerf \subseteq I$, the converse is hold.

Proof. $i) - ii)$ It is clear.

$iii)$ Let $f(J) \in f(\mathbb{I}_R(I))$ and $IJ \subseteq I$. Thus $f(IJ) = f(I)f(J) \subseteq f(I)$ and since $f(I)$ is a left ideal of S, then $f(J) \in \mathbb{I}_S(f(I))$. For the converse, let $f(J) \in \mathbb{I}_S(f(I))$. Thus $f(I)f(J) \subseteq f(I)$. Take $x \in I$ and $z \in J$. There is an element $y \in I$ such that $f(xz) = f(y)$ and so $f(xz-y) = 0$ and $xz - y \in Kerf \subseteq I$ and $xz \in I$. Thus $IJ \subseteq I$ and $f(J) \in f(\mathbb{I}_R(I))$. □

It is clear that if I is a prime ideal, then I is a left O-prime ideal. The following Lemma 2.4 shows that the converse does not hold.

Lemma 2.4. *Any maximal left ideal of a ring is a left O-prime ideal.*

Proof. Let P be a maximal left ideal and I, J be left ideals such that $PJ \subseteq P$. If both I and J are not in P, then $P+I = R$ and $P+J = R$. Then $R = (P+I)(P+J) = P + IJ$. Therefore, IJ is not in P and so P is a left O-prime ideal of R. □

Lemma 2.5. *Let I be proper ideal of R such that I^2 is a left O-prime ideal of R. Then I is idempotent.*

Proof. Since I^2 is a left O-prime ideal and $I^2I \subseteq I^2$ and $I.I \subseteq I^2$, we get that $I \subseteq I^2$ and $I = I^2$. □

Proposition 2.6. *Let R and S be any rings, $\varphi : R \to S$ an epimorphism and $Ker\varphi \subseteq P$. Then P is a left O-prime ideal of R if and only if $\varphi(P)$ is a left O-prime ideal of S.*

Proof. Let I and J be left ideals of S such that $\varphi(P)J \subseteq \varphi(P)$ and let $IJ \subseteq \varphi(P)$. Then $\varphi^{-1}(I)\varphi^{-1}(J) \subseteq P$ and also $P\varphi^{-1}(J) \subseteq P$. Since P is a left O-prime ideal of R, we get that $\varphi^{-1}(I) \subseteq P$ or $\varphi^{-1}(J) \subseteq P$. Thus either I or J is in $\varphi(P)$ and so $\varphi(P)$ is a left O-prime ideal of R.

Conversely let $\varphi(P)$ be a left O-prime ideal in S. Let AB be in P where A, B left ideals and PB is in P. Thus $\varphi(A)\varphi(B) \subseteq \varphi(P)$ and $\varphi(P)\varphi(B) \subseteq \varphi(P)$. Therefore, either $\varphi(A)$ or $\varphi(B)$ is in $\varphi(P)$. Since $Ker\varphi \subseteq P$, A or B is in P. □

Corollary 2.7. *Let R be a ring. Then P is a left O-prime ideal of R if and only if P/N is a left O-prime ideal of R/N for all $N \subseteq P \subseteq R$.*

Proposition 2.8. *If P is a left O-prime ideal of R and I is a direct summand of R such that $I \subseteq \mathbb{I}_R(P)$, then $I \cap P$ is a left O-prime ideal in I.*

Proof. Let J_1 and J_2 be left ideals of I such that $J_1J_2 \subseteq I \cap P$ and $(I \cap P)J_2 \subseteq I \cap P$. Then $J_1J_2 \subseteq P$ and $PJ_2 \subseteq P$. Since J_1 and J_2 are ideals of R and P is a left O-prime ideal of R, $J_1 \subseteq P$ or $J_2 \subseteq P$ and therefore $J_1 \subseteq I \cap P$ or $J_2 \subseteq I \cap P$. □

We recall that

i) a sequence $\eta(a) = \{a, a_1, ...\}$ is called a sequence of an element a of R if for all $i \in \mathbb{N}$, $a_{i+1} \in a_iRa_i$ and $a_0 = a$,

ii) for a left ideal I, an element a of R is called a strongly nilpotent on I if every sequence of a intersects I. (*i.e.* $\eta(a) \cap I \neq \varnothing$.)

Definition 2.9. *Let I be a left ideal of R. Then $O_R(I)$ is the intersection left O-prime ideals of R containing I and I is a left O-radical if $O_R(I) = I$.*

It is clear that $O_R(I)$ is in the intersection of prime ideals of R containing I since every prime ideal of R is left O-prime. It is obvious that every nilpotent element in commutative ring is a strongly nilpotent element on any ideal of R. Now we use $ON_R(I)$ to denote the left ideal generated by the strongly nilpotent elements on I.

Theorem 2.10. *Let R be any ring and let N, L be left ideals of R. Then $O_R(N) + O_R(L) = R$ if and only if $N + L = R$.*

Proof. Suppose that $O_R(N) + O_R(L) = R$ and $N + L \neq R$. Thus, there exists a left maximal ideal T of R such that $N + L \subseteq T$. Since T is a left O-prime ideal of R, we have $O_R(N) \subseteq T$ and $O_R(L) \subseteq T$. Then

$$O_R(N) + O_R(L) \subseteq T.$$

This is a contradiction. Then $N + L = R$.

Since $N \subseteq O_R(N)$, $L \subseteq O_R(L)$ and $N + L = R$, it follows that

$$O_R(N) + O_R(L) = R.$$

□

Lemma 2.11. *Let I be a left ideal of R and K be any multiplicative set with $I \cap K = \varnothing$. Then there is a left O-prime ideal P of R containing I such that $P \cap K = \varnothing$.*

Proof. Consider the set

$$\Psi = \{L : L \cap K = \varnothing \text{ and } L \text{ is a left ideal of } R\}.$$

By Zorn's lemma, there is a maximal element P in the set Ψ. Let A and B be left ideals such that $PB \subseteq P$. Assume that both A and B are not in P and we prove that AB is not in P. Then both $(P + A) \cap K$ and $(P + B) \cap K$ are not empty. Let $a \in (P + A) \cap K$ and $b \in (P + B) \cap K$. It follows that $ab \in K \cap ((P + A)(P + B))$. This means that AB is not in P and so P is a left O-prime ideal of R. □

Lemma 2.12. *Let I be a left ideal of R and $K = \{a_i \in R : a_0 = a$ and $a_{i+1} \in a_iRa_i$, $i \in \mathbb{N}\}$ be a set of R. If the intersection of I and K is an empty set, then there is a left O-prime ideal P of R containing I such that $P \cap K = \varnothing$.*

Proof. Consider the set

$$\Psi = \{L : L \cap K = \varnothing \text{ and } L \text{ is a left ideal of } R\}.$$

By Zorn's lemma, there is a maximal element P in the set Ψ. Let A and B be left ideals such that $PB \subseteq P$. Assume that both A and B are not in P and we prove that AB is not in P. Then both $(P + A) \cap K$ and $(P + B) \cap K$ are not empty. Let $r_n \in (P+A)\cap K$ and so $r_t \in (P+A)\cap K$ for all $t \geq n$. Similarly, let $r_m \in (P+B)\cap K$ and so $r_v \in (P + B) \cap K$ for all $v \geq m$. Now assume that $n \leq m$. We observe that $r_{m+1} = lr_nkr_m$ for some $l, k \in R$ and so r_{m+1} in $K \cap ((P + A)(P + B))$. Therefore, AB is not in P and so P is a left O-prime ideal of R. □

Lemma 2.13. *Let I be a left ideal of R. Then $O_R(I) \subseteq ON_R(I)$.*

Proof. Let a_i be in $O_R(I)$ but not be a strongly nilpotent element on I. Then there is a sequence $K = \{a_i \in R : a_0 = a$ and $a_{i+1} \in a_iRa_i$, $i \in \mathbb{N}\}$ such that $I \cap K = \varnothing$. Then there is a left O-prime ideal P of R containing I such that $P \cap K = \varnothing$. This is a contradiction with $a_i \in O_R(I)$. □

Lemma 2.14. *Let I be a left ideal of R. Then $ON_R(I) = O_R(I)$ if one of the following conditions holds;*

1) *$axa \notin P$ whenever $xa \notin P$ where P is left O-prime.*
2) *Every left O-prime ideal P which is not ideal is a maximal left ideal.*

Proof. It is enough to show that $ON_R(I) \subseteq O_R(I)$.

Let $a \in ON_R(I)$ but not in $O_R(I)$. Then there is a left O-prime ideal P of R containing I such that a is not in P. For a left O-prime ideal P, we have two cases:

a) Let $PRa \subseteq P$. Since aRa is not in P, there is a non zero element $a_1 = at_0a \in aRa$ but not in P. Then $PRa_1 \subseteq PRa \subseteq P$ and so we get that a_1Ra_1 is not in P, hence there is a nonzero element $a_2 = a_1t_1a_1 \in a_1Ra_1$. By using this method, we get the sequence $\eta(a)$ of a is the set $\eta(a) = \{a_i : a_{i+1} \in a_iRa_i$ and $a_0 = a$, $i \in \mathbb{N}\}$ but $\eta(a)$ does not contain any element of I since for all $i \in \mathbb{N}$, $a_i \notin P$. Therefore a is not a strongly nilpotent element of R on I, a contradiction.

b) Let $PRa \subsetneq P$.

i) Let the condition in (1) hold. There are elements $p_0 \in P$ and $x \in R$ such that $(p_0x)\, a \notin P$ and so choose $a_1 = a\,(p_0x)\, a \notin P$ by the condition (1).

ii) Let the condition in (2) hold. Then P is a maximal left ideal of R and $P + PRa = R$. Hence $1 = m + ka$ for some $m \in P$ and $k \in PR$ and so $a - am = aka \notin P$. Now choose $a_1 = aka$.

If $PRa_1 \subseteq P$, then using the argument in (a), we may choose an element $a_2 = a_1ta_1 \notin P$ where $t \in R$.

If $PRa_1 \nsubseteq P$, then following the procedure in (b) for a_1, we may get an element $a_2 = a_1ta_1 \notin P$ where $t \in R$.

Therefore, we have the sequence $\eta(a)$ of a as the set $\eta(a) = \{a, a_1, a_2, ... : a_{i+1} \in a_iRa_i$ and $a_0 = a,\ i \in \mathbb{N}\}$ but $\eta(a)$ does not contain any element of I since for all $i \in \mathbb{N}$, $a_i \notin P$. Therefore a is not a strongly nilpotent element of R on I. □

The following example shows that there is a left O-prime ideal satisfying the condition in (1).

Example 2.15. *Let* $R = \begin{bmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{bmatrix}$ *be a ring where* F *is a field. Then* $P = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & 0 \end{bmatrix}$

is a left ideal but not a right ideal. Let us compute the left ideal $\mathbb{I}_R(P) = \{a \in R : PRa \subseteq P\}$.

If $q = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ *is in* $\mathbb{I}_R(P)$*, then* $Pq \subseteq PRq \subseteq P$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & e \\ 0 & 0 & 0 \end{bmatrix} \in P$$

if and only if $e = 0$. *Thus* $q = \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix}$ *and so*

$$q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & 0 & f \end{bmatrix}$$

Hence $q_1 \in Rq$ *and so* $q_1 \in \mathbb{I}_R(P)$ *since* $PRq_1 \subseteq PRq \subseteq P$. *Therefore,* $f = 0$ *and so* $q \in P$. *This means that* $\mathbb{I}_R(P) = P$ *and so* P *is a left O-prime ideal.*

Let $g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. *Then both* xg *and* *pxg are not in P and so this means that PRg is not in P. Also we get that gpxg is not in P. Therefore, the condition (1) in Lemma 2.14 is not satisfied in general. If* $g_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \notin P$, $g_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin P$, *then* $g_2 g_1 \notin P$ *but* $g_1 g_2 g_1$ *is in* P.

Theorem 2.16. *Let P be a left ideal of R, which is not a right ideal. Suppose that P is maximal among all left ideals in R that are not finitely generated. Then P is a left O-prime ideal of R.*

Proof. Suppose that $\mathbb{I}_R(P) \neq P$ and so $b \in \mathbb{I}_R(P) - P$. Let a be in R such that aRb is in P. Then $P + Rb$ is different from P and $P + Rb$ is finitely generated. Let $\{p_1 + r_1 b, ..., p_t + r_t b\}$ be a generating set for $P + Rb$ where $p_i \in P$ and $r_i \in R$.

Define the set $K = \{y \in R : yb \in P\}$. Then clearly, K is a left ideal containing both P and a. Assume that a is not in P. Otherwise, P is a left O-prime ideal. Hence K is a finitely generated left ideal of R since $P + Ra \neq P$.

Take an element x in $P \subsetneqq P + Ra \subseteq K$. Since K is finitely generated, we get $x = u_1(p_1 + r_1 b) + ... + u_t(p_t + r_t b)$ for some $u_i \in R$ and so

$$x - (u_1 p_1 + ... + u_t p_t) = (u_1 r_1 + ... + u_t r_t)b$$

Hence $(u_1 r_1 + ... + u_t r_t) \in K$. This means that $x \in Rp_1 + ... + Rp_t + Kb$ and so $P = Rp_1 + ... + Rp_t + Kb$ which implies that P is finitely generated, a contradiction. □

Corollary 2.17. *If every left O-prime ideal which is not a right ideal in a ring R is finitely generated, then R satisfies ascending chain condition on left ideals which are not right ideals.*

Proof. Let every left O-prime ideal in a ring R be finitely generated. Define the set $\Omega = \{I_i : I_i \text{ is a left ideal of } R \text{ but not finitely generated}\}$. $\Omega \neq \oslash$, $J = \cup I_i$ is not a finitely generated ideal in R and J is the upper bound in the set Ω. By Zorn's Lemma, there is a maximal element P in the set Ω. By Theorem 2.16, P is a left O-prime ideal of R and then R satisfies ascending chain condition on left ideals which are not right ideals. □

This leads to a noncommutative generalization of Cohen's Theorem for left O-prime ideals.

Corollary 2.18. *(A noncommutative Cohen's Theorem for left O-prime ideals) If every left O-prime ideal in a ring R is finitely generated, then R is a left Noetherian ring.*

Theorem 2.19. *Let R be a noncommutative ring satisfying the ascending chain condition on left O-radical ideals. Then any left O-radical ideal in R is the intersection of a finite number of left O-prime ideals. In particular any left ideal in R is the intersection of a finite number of left O-prime ideals.*

Proof. If not, let a left ideal I be maximal among those for which the assertion fails. Clearly, I is not a left O-prime ideal and so $\mathbb{I}_R(I) \neq I$. Take $a \in R - I$ and $b \in \mathbb{I}_R(I) - I$ with $aRb \subseteq I$. Let J be a left O-radical of $I + Ra$ and K a left O-radical of $I + Rb$. Since I is maximal, J and K are each expressible as a finite intersection of left O-prime ideals. We reach a contradiction proving that $I = J \cap K$.

Let $x \in J \cap K$. Then x is a strongly nilpotent element on both $I + Ra$ and $I + Rb$. If $T = \{a_i : a_{i+1} \in a_i R a_i$ and $a_0 = x,\ i \in \mathbb{N}\}$, then there exits $a_n \in (I + Ra) \cap T$ and so $a_t \in (I + Ra) \cap T$ for all $t \geq n$. Similarly, there exists $a_m \in (I + Rb) \cap T$ and so $a_v \in (I + Rb) \cap T$ for all $v \geq m$ for some $n, m \in \mathbb{N}$. Now assume that $n \leq m$. Then we observe that $a_{m+1} = l a_n k a_m \in T$ for some $l, k \in R$ and so a_{m+1} in $T \cap ((I + Ra)(I + Rb)) = T \cap I$. Therefore, x is in a left O-radical of I and so in I. □

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References

[1] M. Alkan and Y. Tıraş, On prime submodules, Rocky Mountain J. Math. 37 (3) (2007), 709–722.
[2] M. Alkan and Y. Tıraş, Projective modules and prime submodules, Czechoslovak Math. J. 56 (131) (2006), 601–611.
[3] M.L. Reyes, One-sided prime ideals in noncommutative algebra, Thesis of Doctor of Philosophy in Mathematics, University of California, Berkeley. (2010).
[4] W. Anderson and K.R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York-Heidelberg-Berlin, 1992.
[5] B.J. Gardner, and R. Wiegandt, Radical Theory of Rings, Marcel Dekker, Inc., New York-Basel-Berlin, 2004.
[6] S. Çeken, and M. Alkan, On Prime Submodules And Primary Decompositions In Two-Generated Free Modules, Taiwan. Jour. Math. 17 (2013), 133-142.
[7] S. Çeken, and M. Alkan, On τ–extending modules, Med.J.Math. 9, (2012), 129-142
[8] D. S. Dummit, and R. M. Foote, Abstract Algebra, Prentice Hall, Upper Saddle River, N. J. 1999.
[9] J. Dauns, Prime modules, J. Reine Angew Math. 298 (1978), 156-181.
[10] T. Y. Lam, A First Course in Noncommutative Rings, Springer, 2001.
[11] J.C. McConnel, and J.C. Robson, Noncommutative Noetherian Rings, Wiley Chichester 1987.
[12] P. F. Smith, Radical submodules and uniform dimension of modules, Turk J. Math. 28 (2004), 255-270.
[13] J. Dauns, Prime modules and one–sided ideals in ring Theory and Algebra III' (Proceeding of the Third Oklahoma Conference), B.R. McDonald (editor) (Dekker, New York 1980), 301-344.
[14] K.H. Leung and H. S. Man, On Commutative Noetherian Rings which satisfy the radical formula, Glasgow Math J. 39 (1997), 285–293.
[15] Y. Tiras and M. Alkan, Prime modules and submodules, Comm. in Algebra 31 (2003), 395-396.
[16] Irving Kaplansky, Commutative Rings, revised ed., The University of Chicago Press, Chicago, Ill.-London, 1974.
[17] I. S. Cohen, Commutative rings with restricted minimum condition, Duke Math. J. 17 (1950), 27–42.

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